Probabilistic Methods in Combinatorics

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Assignment 1

To solve for the Example class on 25th February. Submit the solution of Problem 1 by Sunday 23rd February if you wish feedback on it.

The solution of each problem should be no longer than one page!

Starred problems are typically harder. Don't worry if you cannot solve them.

Problem 1. Recall that the Ramsey number $R(k,t)$ is the smallest integer n such that no matter how you colour the edges of the complete graph K_n with red and blue there is always either a complete red clique on k vertices or a complete blue clique on t vertices. Prove that if there is a real p, $0 \leq p \leq 1$, such that:

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$$

then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$. Using this, show that the following holds, for some constant c.

$$R(4,t) \geq c \cdot \frac{t^{3/2}}{(\log t)^{3/2}}.$$

Problem 2. Let $\mathcal{A}$ be a family of subsets of $[n] := \{1,\dots,n\}$ such that there are no distinct $A, B \in \mathcal{A}$ with $A \subseteq B$. Prove that

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Hint 1: Consider a random permutation σ of $[n]$.
Hint 2: Compute the expectation of the random variable X defined by

$$X(\sigma) = |\{i \in [n] : \{\sigma(1), \dots, \sigma(i)\} \in \mathcal{A}\}|.$$

Problem 3. Recall that a tournament T is said to have the property S_k if for every set of k vertices there is one which dominates them all. In the lectures you saw that $f(k) = O(k^2 2^k)$, where $f(k)$ denotes the minimum size of a tournament with property S_k. The aim of this exercise is to give lower bounds on $f(k)$.

(a) Prove that $f(k) \geq 2^{k+1} - 1$ for any $k \in \mathbb{N}$.

(b)* Prove that $f(k) = \Omega(k2^k)$.